Quasi-stationary distribution for R-recurrent Markov chains

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Discrete-time Markov chain

1. Let $(X_n)_{n\in\mathbb{Z}_+}$ be a Markov chain with the countable state space E:

$$\mathbb{P}[X_n = j|X_{n-1} = i, \cdots] = \mathbb{P}[X_n = j|X_{n-1} = i] =: p_{ij}.$$

   n-step transition probability

$$p_{ij}^{(n)} = \mathbb{P}[X_n = j|X_0 = i].$$

2. Assume $P = (p_{ij})_{i,j\in E}$ is irrreducible and aperiodic.
3. For $H \subset E$, define the return time :

$$\tau_H^+ = \inf\{n \geq 1 : X_n \in H\}.$$

   Denote $\tau_j^+ = \tau_H^+$ when $H = \{j\}$.
4. For $n \geq 1$,

$$f_{ij}^{(n)} = \mathbb{P}_i[\tau_j^+ = n], \quad f_{ij} = \mathbb{P}_i[\tau_j^+ < \infty].$$

Positive recurrence, stationary distribution, ergodicity

1. j is recurrent iff $f_{jj} = 1$.
2. If j is recurrent, it is positive recurrent iff $\mathbb{E}_j \tau_j^+ < \infty$.
3. Stationary distribution:

$$\pi_j = \frac{1}{\mathbb{E}_j \tau_j^+}.$$

4. j ergodic:

$$p_{ij}^{(n)} \to \pi_j = \frac{1}{\mathbb{E}_j \tau_j^+} > 0.$$

5. Equivalence between positive recurrence, stationary distribution and ergodicity.

[2]M.-F.Chen, Y.-H.Mao(2021). Introduction to stochastic processes.

λ-QSD

P on a countable set $E \cup \{0\}$, $p_{00} = 1$, $\tau_0 = \inf\{n \geq 0 : X_n = 0\}$. Assume P is irreducible on E and $\mathbb{P}_i[\tau_0 < \infty] = 1$. $\forall\, i \in E$, $\exists\, \lambda > 0$ and proper probability $u = u_\lambda$ on E such that

$$uP = \lambda u \quad \text{on } E$$

or

$$\sum_{i \in E} u_i p_{ij} = \lambda u_j, \quad \forall\, j \in E.$$

Then u is called a λ-QSD.

[6]Yaglom(1947). Certain limit theorems of the theory of branching random processes.

[5]Erik A van Doorn and Philip K Pollett(2012). Quasi-stationary distributions for discrete-state models.

Convergence rate of P

1. $$R = \inf\{r : \sum_{n\geq 0} r^n p_{ij}^{(n)} = \infty\} = \sup\{r : \sum_{n\geq 0} r^n p_{ij}^{(n)} < \infty\}.$$

2. $R \geq 1$, independent of i, j.
   - $R = 1$: identical to the usual concepts of recurrence and transience.
   - $R > 1$: a subclassfication of transient classes.

1. Quasi-stationary distribution(QSD)
2. R-recurrence (positivity)
3. Representation of QSD by a new chain

R-recurrence

1. $P_{ij}(R) = \sum_{n=0}^{\infty} R^n p_{ij}^{(n)} = \infty$ iff $F_{jj}(R) = \sum_{n=1}^{\infty} R^n f_{jj}^{(n)} = 1$.
2. $\Leftrightarrow$ R-invariant measure $u > 0$ and vector $v > 0$ are unique up to constant multiples,
$$uP = (1/R)u, \qquad Pv = (1/R)v.$$
3. If P is R-recurrent, then P is R-positive iff $\sum_{i\in E} u_i v_i < \infty$, and then as $n \to \infty$,
$$R^n p_{ij}^{(n)} \to \frac{v_i u_j}{\sum\limits_{i\in E} u_i v_i}.$$
4. Relation between QSD, R-recurrence, R-positivity.

[4]Kingman,J.F.C.(1963).The exponential decay of Markov transition probabilities.

Questions

1. What are explicit representations of u and v?
2. $(1/R)$-QSD: $\sum_{i\in E} u_i < \infty$?
3. R-positivity: $\sum_{i\in E} u_i v_i < \infty$?

Taboo probability

Let H be an arbitrary set of states. We define

$$ {}_Hp_{ij}^{(n)} = \mathbb{P}[X_n = j, X_v \notin H, 1 \le v \le n-1 | X_0 = i], \quad n \ge 1. $$

1. H is singleton $\{k\}$: ${}_kp_{ij}^{(n)}$.
2. Taboo set: $H \cup \{k\}$, ${}_{k,H}p_{ij}^{(n)}$.
3. $f_{ij}^{(n)} = {}_jp_{ij}^{(n)}$, ${}_Hf_{ij}^{(n)} = {}_{j,H}p_{ij}^{(n)}$. Particularly if $j \in H$, ${}_Hf_{ij}^{(n)} = {}_Hp_{ij}^{(n)}$.
4. If $i \notin H$, ${}_Hp_{ij}^{(0)} = \delta_{ij}$, if $i \in H$, ${}_Hp_{ij}^{(0)} = 0$.

[3]Chung (1967) Markov Chains With Stationary Transition Probabilities.

Single-exit case

1. Let $P = (p_{ij})_{i,j\geq 1}$ with $\sum_{j\in E} p_{1j} < 1$ and $\sum_{j\in E} p_{ij} = 1,\ \forall\ i \geq 2.$
2. $$y_i^{(1)} = Rp_{1i},$$ $$y_i^{(n+1)} = R\sum_{j\neq 1} y_j^{(n)} p_{ji},$$
3. $u_i = \sum_{n=1}^{\infty} y_i^{(n)} = R\sum_{j\neq 1} u_j p_{ji} + Rp_{1i} = R\sum_{j\in E} u_j p_{ji}\ (u_1 = 1).$
4. R-recurrence $\Leftrightarrow u_1 = \sum_{n=1}^{\infty} R^n {}_1p_{11}^{(n)} = 1.$

Sum of u

1. We see that

$$
\begin{aligned}
\sum_{i\in E} u_i &= \sum_{i\in E}\sum_{n\geq 1} R^n{}_1p_{1i}^{(n)} \\
&= \sum_{n\geq 1} R^n \mathbb{P}_1[X_n \in E, \tau_1^+ \geq n] \\
&= \sum_{n\geq 1} R^n \left\{ \sum_{m=n}^{\infty} \mathbb{P}_1[\tau_1^+ = m, \tau_0 > n] + \mathbb{P}_1[\tau_1^+ = \infty, \tau_0 > n] \right\} \\
&= \frac{1}{R-1}\mathbb{E}_1 R^{\tau_0} I_{\{\tau_1^+ = \infty\}} = \frac{R}{R-1} p_{10}.
\end{aligned}
$$

2. There is a $(1/R)$-QSD when P is single-exit.

Explicit representations of u and v

Theorem 1

Assume P is irreducible and aperiodic on E, 0 is certainly absorbing, $R > 1$, R-recurrent. Let $H = \{i \in E : p_{i0} > 0\}$ such that $|H| < \infty$, fix $k \in H$, then $\forall\, j \in E$,

$$u_j = \sum_{i \in H} u_i \sum_{n=1}^{\infty} R^n{}_H p_{ij}^{(n)}, \quad v_j = \sum_{i \in H} \sum_{n=1}^{\infty} R^n{}_H f_{ji}^{(n)} v_i$$

satisfy $uP = (1/R)u, \; Pv = (1/R)v$ and

$$\frac{(R-1)u}{\sum_{i \in H} u_i \mathbb{E}_i R^{\tau_0} I_{\{\tau_H^+ = \infty\}}}$$

is a $(1/R)$-QSD of P, where $\forall\, j \in H$, $u_j = \sum_{n \geq 1} R^n{}_k p_{kj}^{(n)}, \quad v_j = \sum_{n \geq 1} R^n f_{jk}^{(n)}$.

R-positive recurrence

Furthermore, P is R-positive recurrent if and only if $\mathbb{E}_k \tau_k^+ R^{\tau_k^+} < \infty$, and then

$$R^n p_{ij}^{(n)} \to \frac{v_i u_j}{\mathbb{E}_k \tau_k^+ R^{\tau_k^+}}, \quad n \to \infty.$$

- If $R = 1$, then

$$p_{ij}^{(n)} \to \frac{u_j}{\mathbb{E}_k \tau_k^+}, \quad n \to \infty,$$

where $u_j = \sum_{n=1}^{\infty} {}_k p_{kj}^{(n)}$. Particularly,

$$p_{ik}^{(n)} \to \frac{1}{\mathbb{E}_k \tau_k^+}, \quad n \to \infty.$$

h-transform

1. Let P be irreducible and aperiodic.
2. Assume $H = \{i \in E : p_{i0} > 0\} = \infty$.
3. Let the harmonic function $h = (h_i)$,

$$Ph(i) = h_i, \ \forall \ i \neq 1; \ h_1 = 1,$$

$\tau_1 = \inf\{n \geq 0 : X_n = 1\}, \ h_i = \mathbb{P}_i[\tau_1 < \infty].$

4. Define $P^h = (p^h_{ij})$ by

$$p^h_{ij} = \frac{p_{ij} h_j}{h_i}, \quad \forall \ i, \ j.$$

Then $\sum_{j \in E} p^h_{ij} = 1, \ \forall \ i \neq 1$, and $\sum_{j \in E} p_{1j} < 1$, i.e. P^h is single-exit.

h-transform

1. We see that

$$u_i^h = \sum_{n=1}^{\infty} R^n {}_1p_{1i}^{h(n)} = \sum_{n=1}^{\infty} R^n {}_1p_{1i}^{(n)} h_i$$

satisfies $\sum_{i\in E} u_i^h p_{ij}^h = (1/R)u_j^h$ or $\sum_{i\in E} \frac{u_i^h}{h_i} p_{ij} = (1/R)\frac{u_j^h}{h_j}$.

2. $u_i = \sum_{n=1}^{\infty} R^n {}_1p_{1i}^{(n)}$ is the R-invariant measure of P.
3. $\inf_{i\in E} \mathbb{P}_i[\tau_j < \infty] > 0$ for some $j \in E \Rightarrow$
4. $(u_i)_{i\in E}$ is a $(1/R)$-QSD of P.

Infinite-exit case

Theorem 2

Assume P is irreducible on E and 0 is certainly absorbing and R-recurrent. Let $H = \{i \in E : p_{i0} > 0\}$. Assume $\inf_{i\in E} \mathbb{P}_i[\tau_j < \infty] > 0$ for some j, then

$$u_i = \sum_{n=1}^{\infty} R^n {}_1p_{1i}^{(n)}$$

satisfies

$$uP = (1/R)u$$

and

$$\frac{(R-1)\sum_{n=1}^{\infty} R^n {}_1p_{1i}^{(n)}}{\sum_{i\in H} u_i \mathbb{E}_i R^{\tau_0} I_{\{\tau_H^+ = \infty\}}}$$

is a QSD of P.

Perron-Frobenius theorem

Perron-Frobenius theorem

For a non-negative, irreducible matrix A, the largest eigenvalue ρ is positive, its corresponding left eigenvector u and right eigenvector v are positive as well, that is

$$\begin{cases} uA = \rho u, \\ Av = \rho v. \end{cases}$$

- Assume that $\sum u_i = 1$ and $\sum u_i v_i = 1$.

A clever argument due to Cerf and Dalmau

1. A is a primitive matrix of size N,
2. Let u be Perron-Frobenius (left) eigenvector:

$$uA = \lambda u, \quad u > 0, \lambda > 0,$$

3. Set $\sum_j a_{ij} = f(i)$, $M_{ij} = \frac{a_{ij}}{f(i)}$,
4. let $(X_n)_{n\in\mathbb{Z}_+}$ be a Markov chain with state space $\{1, \cdots, N\}$ and transition matrix M, $\tau_j^+ = \inf\{n \geq 1 : X_n = j\}$.

[1]Cerf and Dalmau(2017)A Markov chain representation of normalized Perron-Frobenius eigenvector.

Representation of u

Theorem (Cerf and Dalmau(2017))

Let $1 \leq k \leq N$. The normalized Perron–Frobenius eigenvector u of A is given by the formula

$$\forall\ i \in \{1, \cdots, N\},\ u_i = \frac{\mathbb{E}_k\left(\sum_{n=0}^{\tau_k^+ - 1} \left(I_{\{X_n = i\}} \lambda^{-n} \prod_{m=0}^{n-1} f(X_m)\right)\right)}{\mathbb{E}_k\left(\sum_{n=0}^{\tau_k^+ - 1} \left(\lambda^{-n} \prod_{m=0}^{n-1} f(X_m)\right)\right)}.$$

- If A is stochastic, then $\lambda = 1$ and $f \equiv 1$, to derive that $\pi_i = \frac{1}{\mathbb{E}_i \tau_i^+}$.

Countable Matrix

1. Let $A = (a_{ij})_{i,j \in \mathbb{Z}_+}$ be irreducible and aperiodic,
2. Assume that $\forall\ i \in \mathbb{Z}_+$,

$$f(i) = \sum_j a_{ij} < \infty.$$

3. What we can do Perron-Frobenius for A whether we have $uA = \rho u,\ \rho > 0,\ u > 0$?

Convergence rate

1. Define the convergence rate

$$\rho = \inf\{\lambda > 0 : \sum_{n\geq 0} \lambda^n a_{ij}^{(n)} = \infty\}$$
$$= \sup\{\lambda > 0 : \sum_{n\geq 0} \lambda^n a_{ij}^{(n)} < \infty\}.$$

2. By irreducibility, ρ is independent of i, j.
3. ρ is critical in the sense that $\sum_n \rho^n a_{kk}^{(n)}$ can be finite or infinite.

Main idea

1. Define: $\forall\ i,\ j,$

$$b_{ij}^{(1)} = a_{ij},\ b_{ij}^{(n)} = \sum_{k\neq i} b_{ik}^{(n-1)} a_{kj},\ n \geq 2.$$

2. $\forall\ k \in E,$

$$y_i^{(1)} = \rho a_{ki},\ y_i^{(n+1)} = \rho \sum_{j\neq k} y_j^{(n)} a_{ji},$$

3. Minimal nonnegative solution:

$$\begin{aligned} u_i = \sum_{n=1}^{\infty} y_i^{(n)} &= \rho \sum_{n=2}^{\infty} \sum_{j\neq k} y_j^{(n-1)} a_{ji} + \rho a_{ki} \\ &= \rho \sum_{j\neq k} u_j a_{ji} + \rho a_{ki}. \end{aligned}$$

4. Key: $u_k = \sum_{n=1}^{\infty} \rho^n b_{kk}^{(n)} = 1.$

Recurrence

- Introduce two generation functions:

$$A_{kk}(s) = \sum_{n=0}^{\infty} a_{kk}^{(n)} s^n, \quad B_{kk}(s) = \sum_{n=1}^{\infty} b_{kk}^{(n)} s^n, \quad s \in (0, \rho),$$

- $B_{kk}(s) = 1 - 1/A_{kk}(s)$.

Lemma

Let $k \in E$. *Assume that* $\rho \in (0, \infty)$ *and* $\sum_{n=0}^{\infty} \rho^n a_{kk}^{(n)} = \infty$, *then*

$$1 = \sum_n b_{kk}^{(n)} \rho^n.$$

Representation of u

Theorem 3

Fix $k \in E$. Assume $f(i) < \infty,\ \forall\ i \in E,\ \rho > 0$ and $\sum_{n=0}^{\infty} \rho^n a_{kk}^{(n)} = \infty$.

Then

$$i \in E,\ u_i = \mathbb{E}_k \left(\sum_{n=0}^{\tau_k^+ - 1} \left(I_{\{X_n = i\}} \rho^n \prod_{m=0}^{n-1} f(X_m) \right) \right) \in (0, \infty)$$

and $u = (u_i)_{i \in E}$ satisfies

$$uA = (1/\rho)u.$$

Corollary

Corollary

We note that

$$\sum_{i\in E} u_i = \mathbb{E}_k\left(\sum_{n=0}^{\tau_k^+-1}\left(\rho^n\prod_{m=0}^{n-1}f(X_m)\right)\right).$$

To assure that $(u_i)_{i\in E}$ is summable, we shall assume that for some k, $\mathbb{E}_k\left(\sum_{n=0}^{\tau_k^+-1}\left(\rho^n\prod_{m=0}^{n-1}f(X_m)\right)\right)<\infty$.

Finite Markov chain

1. Let P be an irreducible and aperodic, sub-stochastic transition probability matrix.
2. By Perron-Frobenius theorem, $\exists\ \rho > 0,\ u > 0,$
$$uP = \rho u.$$
3. By assuming $\sum_{i\in E} u_i = 1$, we see that u is a QSD for P.
4. Now, Cerf and Dalmau theorem gives an elegant representation of QSD.

Countable Markov chain

Theorem 4

Assume P is irreducible on E, $R > 1$, 0 is certainly absorbing and R-recurrent. Let $H = \{i \in E : p_{i0} > 0\}$ such that $|H| < \infty$. Then

$$u_j = \sum_{i \in H} u_i \mathbb{E}_i \left(\sum_{n=0}^{\widetilde{\tau}_H^+ - 1} I_{\{\widetilde{X}_n = j\}} R^n \prod_{m=0}^{n-1} f\left(\widetilde{X}_m\right) \right), \quad j \in E$$

satisfies $uP = (1/R)u$ and μ is a QSD of P, where $k \in H$ is fixed,

$$u_j = \sum_{n=1}^{\infty} R^n {}_k p_{kj}^{(n)}, \quad j \in H; \quad \mu = \frac{R-1}{R \sum_{i \in H} u_i p_{i0}} u.$$

Single-exit case

Particularly if H is singleton $\{1\}$(say), then

$$u_1 = 1,$$
$$u_i = \mathbb{E}_1 \left(\sum_{n=0}^{\widetilde{\tau}_1^+ - 1} \left(I_{\{\widetilde{X}_n = i\}} R^n \prod_{m=0}^{n-1} f(\widetilde{X}_m) \right) \right)$$

satisfies $uP = (1/R)u$ and μ is a QSD of P.

h-transform

Theorem 5

Assume P is irreducible on E and 0 is certainly absorbing and $R > 1$, R-recurrent. Let $H = \{i \in E : p_{i0} > 0\}$. Assume $\inf_{i \in E} \mathbb{P}_i[\tau_j < \infty] > 0$ for some j, then

$$u_i = \mathbb{E}_1 \left(\sum_{n=0}^{\widetilde{\tau}_1^+ - 1} \left(I_{\{\widetilde{X}_n = i\}} R^n \prod_{m=0}^{n-1} f(\widetilde{X}_m) \right) \right)$$

satisfies $uP = (1/R)u$ and

$$\frac{\mathbb{E}_1 \left(\sum_{n=0}^{\widetilde{\tau}_1^+ - 1} \left(I_{\{\widetilde{X}_n = i\}} R^n \prod_{m=0}^{n-1} f(\widetilde{X}_m) \right) \right)}{\mathbb{E}_1 \left(\sum_{n=0}^{\widetilde{\tau}_1^+ - 1} \left(R^n \prod_{m=0}^{n-1} f(\widetilde{X}_m) \right) \right)}$$

is a QSD of P.

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Thanks for your attention!